EURODOLLAR FUTURES CONVEXITY ADJUSTMENTS IN STOCHASTIC VOLATILITY MODELS

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Abstract. A formula that explicitly incorporates volatility smile, as well as a realistic correlation structure of forward rates, in computing Eurodollar futures convexity adjustments is derived. The effect of volatility smile on convexity adjustments is studied and is found significant.

1. Introduction

An interest rate term curve (a collection of discount factors for all future dates) is the fundamental input for interest rate derivatives valuation. A term curve is created from (or fit to) prices of liquidly-traded interest rate instruments. One of the most liquid interest rate contracts is the Eurodollar futures (or ED) contract (and its Euro and Yen equivalents). These are futures contracts with the underlying being a Libor rate, and they are traded on the Chicago Mercantile Exchange (their equivalents are traded on CME, SIMEX and LIFFE). ED futures contracts are fundamental building blocks of interest rate term curves.

The ED contract is subject to margining\(^1\). Its price not only incorporates information on the relevant forward rate (the input necessary for building an interest rate term curve), but also on what is known as a *convexity adjustment*. The convexity adjustment is the extra value that a futures contract on a rate has over a forward contract on the same rate, arising from the fact that the profits can be reinvested daily at a higher rate, while the losses can be financed at a lower rate. The value assigned to the convexity adjustment is model-dependent, that is, it depends on a model of future evolution of interest rates. In order to estimate the relevant forward rate for a given period from the ED contract price, this convexity adjustment needs to be estimated first.

In this paper we present a new formula for computing convexity adjustments. It is different from what has been proposed previously (see Section 2) in a number of important ways. The most fundamental difference is that our formula incorporates the volatility smile, unlike any other formula proposed to date. We demonstrate that accounting for volatility smile properly is vital to value the convexity adjustment correctly\(^2\).

Let us define \( f(t, T, S) \) to be the forward value, at time \( t \), of a simple rate covering the period \([T,S]\), with \( 0 \leq t \leq T < S \); that is,

\[
f(t, T, S) = \frac{P(t, T) - P(t, S)}{(S-T)P(t, S)},
\]

where \( P(\cdot, \cdot) \) are zero-coupon discount factors. As well-known, the value of a futures contract on an asset equals the expected value of its payoff under the risk-neutral measure. In particular, the value at time \( t \) of the futures contract on the rate \( f(\cdot, T, S) \) is given by (as in [HK00])

\[
F(t, T, S) = \mathbb{E}_t^Q f(T, T, S),
\]

where \( Q \) is the risk-neutral measure.

The convexity adjustment is defined as the difference between the futures and the forward on the rate,

\[
\mathbb{E}_t f(T, T, S) - f(t, T, S).
\]
Thus, conceptually, computing the convexity adjustment should be as simple as evaluating (1.1) using a Monte-Carlo simulation in an interest rate model of choice. This is, however, unfeasible in practice due to speed constraints. We address this issue by deriving a closed-form approximation to the “true” model value of the convexity adjustment, an approximation that is very accurate and fast to compute.

For the same practical reasons, the formula for convexity adjustments should depend on observable market inputs in the most direct way possible, with lengthy model and curve calibrations reduced to a minimum or eliminated altogether. In general, as we will demonstrate in the paper, the values of convexity adjustments depend on the law of a joint evolution of a large collection of rates. We reduce this complex dependence to a small number of parameters.

Furthermore, we separate the parameters into two categories: those that change often, and those that do not. The parameters that change, volatility parameters, are taken directly from the prices of options on ED contracts with different expiries and, importantly, strikes. These parameters can be updated in real time. The other category of parameters, correlation parameters as we call them, come from calibrating a model with a rich volatility structure to caps and swaptions. While the latter parameters cannot be updated in real time, they do not need to be. They can be updated on an infrequent basis and kept constant between updates.

In summary, our formula for ED convexity adjustments,

- Expresses forward rates in terms of futures rates (not the other way around);
- Introduces volatility smile as an explicit and direct input;
- Is designed for speed of valuation;
- Incorporates a sensible correlation structure of rates that can be updated infrequently.

The formula is derived by the following procedure:

1. First, an expansion technique is applied to derive a model-independent relationship that expresses a forward rate as a functional of a collection of futures with expiries on or before the forward rate’s expiry.
2. Second, the variance terms that appear in the formula are separated into parameters that are easily observed in the market and change often (volatility parameters), and those that require calibration to be estimated but do not change often (correlation parameters).
3. Next, the volatility parameters are represented in a number of ways, both in a model-independent way as functions of prices of options on ED futures across strikes, and as closed-form expressions involving volatility smile parameters.
4. Finally, the correlation parameters are expressed in terms of the parameters of a forward Libor model (instantaneous Libor volatilities), properly calibrated to relevant market instruments.

Let us comment on the innovative approach used in the first step of the algorithm. Typically an “inverse” procedure is adopted, in which (approximate) values of futures are expressed in terms of forward rates. This still leaves the problem of solving the derived equations to obtain forward rates from market-observed futures. This step is dispensed with in the algorithm we develop. Moreover, with our approach, it appears that significantly more accurate, and model-independent, expansions can be obtained with fewer terms.

In the final stages of preparing this paper, we became aware of the work by Jäckel and Kawai (see [JK04]) who also consider the problem of computing Eurodollar convexity adjustments in displaced-lognormal forward Libor models. While the object of modeling is the same, the differences in the approaches and techniques are significant. Their approximation is derived by formal asymptotic expansions of the drift of a forward rate in a forward Libor model to an arbitrary order. While we use expansions to the second order only, our approach incorporates stochastic volatility and, furthermore, a different quantity is expanded (see previous paragraph). It is also worth reiterating that our main expansion is model-independent.

The paper is concluded with a number of tests. They demonstrate a significant impact that the volatility smile has on convexity adjustments, and also verify the accuracy of proposed approximations.

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3This is similar to the modern approach for computing constant-maturity-swap and Libor-in-arrears convexity adjustments, but has not yet been extended to ED convexities.
2. Review of existing methods

There are several published works that address future-forward convexity adjustments, with Flesaker in [Fle93], Burghardt and Hoskins in [BH95], Kirikos and Novak in [KN97], Vaillant in [Vai99], Musielo and Rutkowski in [MR97], Hunt and Kennedy in [HK00], and Hull in [Hul02] among them.

The convexity adjustment in [Hul02] is given by the expression \( \frac{1}{2} \sigma_t^2 t_1 t_2 \), where \( \sigma \) is the standard deviation of the short rate in one year, \( t_1 \) the expiration of the contract, and \( t_2 \) is the maturity of the Libor rate. This formula is an approximation to Flesaker’s formula. Flesaker in [Fle93] derived the convexity adjustment by computing the required expectation of a Libor rate under the risk-neutral measure using the Ho-Lee model in a continuous and discrete setup. The formula is extended in [KN97], who derived the convexity adjustment in the Hull-White model.

Both Ho-Lee and Hull-White models are Gaussian Heath-Jarrow-Morton (HJM) models, ie HJM models with deterministic bond (or instantaneous forward rate) volatilities. In such models computing convexity adjustments is straightforward. If the bond price satisfies the SDE

\[
dP(t, T) = r(t) P(t, T) dt + \Sigma(t, T) P(t, T) dW(t)
\]

with deterministic \( \Sigma(\cdot, \cdot) \), the ED convexity adjustment \( E^Q[f(T, T, S)] - f(0, T, S) \), where \( f(t, T, S) \) is a simple forward rate for period \( [T, S] \) observed at time \( t \), is equal to

\[
\frac{1 + (S - T) f(0, T, S)}{S - T} \left( e^{\frac{1}{2} \int_0^T (\Sigma(u, S) - \Sigma(u, T))^2 du + \frac{1}{2} \int_0^T ((\Sigma(u, S)) - (\Sigma(u, T))^2) du} - 1 \right).
\]

Flesaker’s formula is obtained by setting \( \Sigma(t, T) \) to \(-\sigma(T - t) \) (Ho-Lee model), and Kirikos-Novak’s formula is obtained by setting \( \Sigma(t, T) \) to \(-\sigma (1 - e^{-\alpha(T-t)}) \) / \( \kappa \) (Hull-White model). These formulas are fast to compute but do not incorporate the volatility smile effect. Moreover, the Ho-Lee formula and the Hull-White formula are too simplistic (being one-factor) to capture the de-correlation effect on convexity adjustments.

Hunt and Kennedy in [HK00] prove from the first principles that if \( H \) is the settlement value of a futures contract, \( H \) is square integrable, and the money market account \( B(t) \) is always positive then the martingale \( E^Q[H] \) is a futures price and that the convexity adjustment is given by

\[
\frac{\text{cov} [H, B(S)]}{P(0, S)},
\]

where the covariance is taken under the risk neutral measure. No model-specific calculations are given.

Vaillant in [Vai99] proposes a model of the form

\[
dF(t, T, S) = \mu(t) F(t, T, S) dt + \sigma_F(t) F(t, T, S) dW(t),
\]

\[
P(t, S) = e^{-(S-t) r(t, S)},
\]

\[
dr(t, S) = \gamma (r_\infty - r(t, S)) dt + \sigma_r(t) r_\infty dZ(t),
\]

\[
d(W(t), Z(t)) = \rho(t) dt,
\]

where \( W, Z \) are standard Brownian motions, \( r(t, S) \) is the spot zero rate, and \( r_\infty \) is some asymptotic value the zero rate reaches. The convexity adjustment \( C(t) = f(t, T, S) / F(t, T, S) \) (defined as the quotient between the forward rate and the futures rate) is given by the formula

\[
C(t) = \exp \left( -r_\infty \int_t^T (S - u) \sigma_r(u) \sigma_F(u) \rho(u) \, du \right).
\]

It is not clear whether his model is arbitrage-free and, moreover, no provision is given for the choice of \( \rho(t) \), a critical input.

Burghardt and Hoskins in [BH95] analyze the market statistically. They look at the discrete effect of mark-to-market and show the existence of the convexity bias empirically. To estimate the convexity bias, they suggest assuming that each forward rate and each zero rate are normally distributed with a given correlation, and to linearize the payoff. The convexity adjustment formula they propose consists of two parts reflecting the two simplifications made.

With few notable exceptions (eg the paper [JK04] mentioned in the introduction), the modeling of ED convexity adjustments has not progressed far beyond simplistic distributional assumptions. In particular, no
model that incorporates a realistic volatility smile has been proposed so far, and no study of the volatility
smile effects on the convexity adjustments has been made. These are the gaps filled by this paper.

3. The convexity formula

3.1. Preliminaries. Let

\[ 0 = T_0 < T_1 < \cdots < T_N, \]
\[ \tau_n = T_{n+1} - T_n, \]

be a tenor structure, i.e., a sequence of (roughly) equidistant dates. We define by \( P(t, T) \) the value, at time
\( t \), of a zero coupon bond paying $1 at time \( T \). We define spanning Libor rates

\[ f_n(t) \equiv f(t, T_n, T_{n+1}) = \frac{P(t, T_n) - P(t, T_{n+1})}{\tau_n P(t, T_{n+1})}, \]
\[ n = 0, \ldots, N - 1. \]

The numeraire \( B_t \) (discretely-compounded money-market account) is defined by

\[ B_{T_0} = 1, \]
\[ B_{T_{n+1}} = B_{T_n} \times (1 + \tau_n f_n(T_n)), \quad 1 \leq n < N, \]
\[ B_t = P(t, T_{n+1}) B_{T_{n+1}}, \quad t \in [T_n, T_{n+1}]. \]

The corresponding measure (known as spot Libor measure) is denoted by \( P \). The \( T_n \)-forward measure (i.e., the measure for which \( P(\cdot, T_n) \) is the numeraire) is denoted by \( P^n \).

Let \( F(t, T, S) \) be the value of a futures contract on the rate \( f(t, T, S) \). We denote \( F_n(t) = F(t, T_n, T_{n+1}) \), i.e., \( F_n(t) \) is the futures contract on the Libor rate \( f_n(t) \).

Proposition 3.1. If the futures contract is marked-to-market only on the dates \( T_0, T_1, \ldots, T_N \), then its value
is given by

\[ F(t, T, S) = E_t f(t, T, S). \]

In particular,

\[ E_0 f_n(T_n) = F_n(0). \]

Proof. See Appendix A.1. \( \blacksquare \)

The following relations hold,

\[ F_n(0) = E_0 f_n(T_n) = E_0 F_n(T_n), \]
\[ f_n(0) = E_0^{n+1} f_n(T_n) = E_0^{n+1} F_n(T_n), \]

for all \( n \). We assume that all \( F_n(0), n = 0, \ldots, N - 1 \), are known; we would like to derive formulas that express forward rates \( \{f_n(0)\} \) in terms of futures \( \{F_n(0)\} \) (and other market-observed quantities).

The following result is straightforward.

Proposition 3.2. For each \( n, n = 1, \ldots, N - 1 \),

\[ f_n(0) = E_0 \left( \prod_{i=0}^{n} \frac{1 + \tau_i f_i(0)}{1 + \tau_i f_i(T_i)} \right) f_n(T_n). \]

This formula expresses the forward rate \( f_n(0) \) as an expectation of a certain payoff under the spot Libor
measure (not forward measure as in (3.2)), the measure that is used in defining futures in (3.1). This formula
is used as a starting point for deriving expressions for convexity adjustments.

Proof. Follows by measure change. \( \blacksquare \)
3.2. Expansion around the futures value. To express the expected value in (3.3) in terms of market-observed quantities, we derive a Taylor series expansion of (3.3) in powers of a small parameter that measures the deviation of each of \( f_n(T_n) \) from its mean in the spot Libor measure, \( F_n(0) = \mathbb{E}_0 f_n(T_n) \).

Fix \( \varepsilon > 0 \), and define \( f_n^\varepsilon \)'s by

\[
f_n^\varepsilon(t) = \varepsilon (f_n(t) - F_n(0)) + F_n(0).
\]

For the future we note that for any \( n \),

\[
(3.4) \quad f_n^0(t) = f_n(t),
\]

\[
(3.5) \quad f_n^0(0) = F_n(0),
\]

\[
(3.6) \quad \frac{\partial f_n^\varepsilon(t)}{\partial \varepsilon} = f_n(t) - F_n(0),
\]

\[
(3.7) \quad f_n^\varepsilon(T_n) = \varepsilon (f_n(T_n) - \mathbb{E}_0 f_n(T_n)) + \mathbb{E}_0 f_n(T_n).
\]

We define

\[
(3.8) \quad V(\varepsilon) = \left[ \prod_{i=0}^n \frac{1 + \tau_i f_i(0)}{1 + \tau_i f_i(T_i)} \right] f_n^\varepsilon(T_n).
\]

It is clear that \( V(1) \) is the value on the right-hand side of (3.3),

\[
(3.9) \quad f_n(0) = \mathbb{E}_0 V(1).
\]

Expanding \( V(\varepsilon) \) into a Taylor series in \( \varepsilon \) yields,

\[
(3.10) \quad V(\varepsilon) = V(0) + \mathbb{E}_0 \frac{\partial V}{\partial \varepsilon}(0) \varepsilon + \frac{1}{2} \mathbb{E}_0 \frac{\partial^2 V}{\partial \varepsilon^2}(0) \varepsilon^2 + O(\varepsilon^3).
\]

The values of the derivatives of \( V(\varepsilon) \) are computed in the following theorem.

**Theorem 3.3.** For any \( n, n = 1, \ldots, N - 1 \),

\[
(3.11) \quad V(0) = \left[ \prod_{i=0}^n \frac{1 + \tau_i f_i(0)}{1 + \tau_i f_i(T_i)} \right] F_n(0),
\]

\[
(3.12) \quad \mathbb{E}_0 \frac{\partial V}{\partial \varepsilon}(0) = 0,
\]

\[
(3.13) \quad \mathbb{E}_0 \frac{\partial^2 V}{\partial \varepsilon^2}(0) = V(0) \sum_{j,m=0}^n D_{j,m} \text{covar}(f_j(T_j), f_m(T_m)),
\]

where the coefficients \( D_{j,m} \) are given by

\[
(3.14) \quad D_{j,m} = \left( -\frac{\tau_j}{1 + \tau_j F_j(0)} + 1_{j=n} \frac{1}{F_n(0)} \right) \left( -\frac{\tau_m}{1 + \tau_m F_m(0)} + 1_{m=n} \frac{1}{F_n(0)} \right)
\]

\[
+ 1_{j=m} \left( \frac{\tau_j^2}{(1 + \tau_j F_j(0))^2} - 1_{j=n} \frac{1}{F_n(0)^2} \right),
\]

and by definition

\[
\text{covar}(f_j(T_j), f_m(T_m)) = \mathbb{E}_0 (f_j(T_j) - \mathbb{E}_0 f_j(T_j))(f_m(T_m) - \mathbb{E}_0 f_m(T_m))
= \mathbb{E}_0 (f_j(T_j) - F_j(0))(f_m(T_m) - F_m(0)).
\]

The proof is given in the appendix. The main contribution of this theorem is that it expresses quantities \( V(0), \mathbb{E}_0 \frac{\partial^2 V}{\partial \varepsilon^2}(0) \) from the series expansion (3.10) in terms of quantities that are either directly observable (i.e., futures and forward rates) or computable (covariances of forward rates). Applying the results of Theorem 3.3 to the representation (3.9) and expansion (3.10), the following corollary follows.
Corollary 3.4. For any \( n, n = 1, \ldots, N - 1 \), the (approximation to) forward rate \( f_{n} (0) \) can be obtained from the futures \( \{ F_{i} (0) \}_{i=0}^{n} \) and forward rates for previous periods \( \{ f_{i} (0) \}_{i=0}^{n-1} \) by solving the following equation,

\[
(3.15) \quad f_{n} (0) = V (0) \left( 1 + \frac{1}{2} \sum_{j,m=0}^{n} D_{j,m} \text{covar} (f_{j} (T_{j}), f_{m} (T_{m})) \right),
\]

with \( V (0) \) and \( D_{j,m} \) given in Theorem 3.3.

This corollary provides an algorithm for solving for forward rates \( f_{n} (0) \) sequentially for all \( n \), using futures prices \( \{ F_{j} (0) \} \) as inputs.

Remark 3.1. The expression on the right-hand side of (3.15) will be simplified in the sections that follow. In many cases the rate to be determined from the expression, \( f_{n} (0) \), will appear on the right-hand side of (3.15) as well. In this case, (3.15) should be treated not just as an identity, but as an equation on \( f_{n} (0) \). While this may seem to complicate the problem of finding \( f_{n} (0) \), in reality the dependence of the right-hand side of (3.15) on \( f_{n} (0) \) is typically mild, and the equation can be solved iteratively in just a few steps.

The formula (3.15) depends on covariances between various forward (or futures) rates. These covariances are not directly observable in the market. We proceed to specify a model that would allow us to relate this quantity to market observables.

By the definition of the covariance,

\[
(3.16) \quad \text{covar} (f_{j} (T_{j}), f_{m} (T_{m})) = \left( \text{var} f_{j} (T_{j}) \text{var} f_{m} (T_{m}) \right)^{1/2} \text{corr} (f_{j} (T_{j}), f_{m} (T_{m})),
\]

where the variances and the correlation are computed under the spot Libor measure.

4. Computing variances of forward rates

4.1. Preliminaries. In this section different approaches to computing the variances of forward (or futures) rates, as required by formula (3.16), are proposed. It is the variances of \( f_{n} (T_{n}) \) under the spot Libor measure that are required by the formula. From the modeling prospective, it is much more natural to compute the variance of each \( f_{n} (T_{n}) \) under its own \( T_{n+1} \)-forward measure, since each \( f_{n} (\cdot) \) is a martingale under its corresponding forward measure. We propose to approximate var’s in (3.16) with \( \text{var}^{n+1} \)’s,

\[
(4.1) \quad \text{var}^{n+1} f_{n} (T_{n}) \approx \text{var}^{n+1} f_{n} (T_{n}),
\]

where by definition

\[
\text{var}^{n+1} f_{n} (T_{n}) \triangleq \mathbb{E}_{0}^{n+1} \left( f_{n} (T_{n}) - \mathbb{E}_{0}^{n+1} f_{n} (T_{n}) \right)^{2}.
\]

It can be shown that the error of this approximation is small\(^4\). With this approximation, the formula for computing forward rates from future ones is expressed as

\[
(4.2) \quad f_{n} (0) = V (0) \left( 1 + \frac{1}{2} \sum_{j,m=0}^{n} D_{j,m} \left( \text{var}^{j+1} f_{j} (T_{j}) \text{var}^{m+1} f_{m} (T_{m}) \right)^{1/2} \text{corr} (f_{j} (T_{j}), f_{m} (T_{m})) \right),
\]

4.2. Model-independent approach. A market in options on ED contracts is very liquid, being perhaps the most liquid market of options on interest rates. The variance of a forward rate can be obtained, in a model-independent way, from the prices of options on the forward rate of different strikes. This is the observation utilized in this section.

A option on a futures contract expires at the same (or close to) time as the futures contract itself. Since at expiration \( F_{n} (T_{n}) = f_{n} (T_{n}) \), the option can be regarded as paying \( (f_{n} (T_{n}) - K)^{+} \) at time \( T_{n+1} \). The value of the option at time 0 is given by\(^5\)

\[
\mathbb{E}_{0}^{B_{T_{n+1}}^{-1}} (f_{n} (T_{n}) - K)^{+} = P (0, T_{n+1}) \mathbb{E}_{0}^{n+1} (f_{n} (T_{n}) - K)^{+}.
\]

\(^4\)This statement can be made rigorous by showing that the two variances agree to a high order in an expansion with respect to a small volatility scaling parameter. The details are beyond the scope of this paper.

\(^5\)Options on ED futures are typically American-style. We treat them as Europeans, which should introduce only a small error.
The forward Libor model with stochastic volatility.

For reasons that will be clear later, a term structure model will be required. A "global" adjustment value depends on prices of options of all strikes, i.e., on the volatility smile.

Stochastic volatility models are widely recognized as providing the superior framework for modeling volatility smiles in interest rates, see [AA02]. Moreover, stochastic volatility models typically have intuitive parameters and efficient numerical methods for valuation and calibration. While only individual forward rates need to be modeled, for reasons that will be clear later, a term structure model will be required. A "global" forward Libor model with stochastic volatility is defined next.

### Proposition 4.1.

The following holds model-independently

\[
\text{var}^{n+1} f_n(T_n) = E_0^{n+1} \left( f_n(T_n) - E_0^{n+1} f_n(T_n) \right)^2 
\]

\[
= 2 \int_{-\infty}^{f_n(0)} E_0^{n+1} (K - f_n(T_n))^+ dK + 2 \int_{f_n(0)}^{\infty} E_0^{n+1} (f_n(T_n) - K)^+ dK.
\]

The proof is standard and is omitted.

This method has an advantage in that, to compute variances in (4.2), it directly uses observable option prices. The forward \( f_n(0) \), the rate to solve for, enters the right-hand side of the equation only as an integration limit in (4.3). The comments of Remark 3.1 apply.

The formula (4.3) demonstrates explicitly one of our main statements, namely that the ED convexity adjustment value depends on prices of options of all strikes, i.e., on the volatility smile.

The formula (4.3) is intuitive and theoretically appealing, but may not be easy to use in practice. This is so because only a discrete set of strikes is typically traded, and not all of them are very liquid. For these reasons, a model-based approach may be preferable. It is based on computing the required variances of forward rates in a model that is calibrated to the volatility smile in options.

4.3. **The forward Libor model with stochastic volatility.** A stochastic variance process \( z(t) \) is defined by the SDE

\[
dz(t) = \theta(z_0 - z(t)) \, dt + \eta \sqrt{z(t)} \, dV(t),
\]

\[
z(0) = z_0.
\]

Let

\[
dW(t) = (dW_1(t), \ldots, dW_K(t))
\]

be a \( K \)-dimensional Brownian motion (under the measure \( Q \)) independent of \( dV \). The Stochastic Volatility forward Libor model is defined by the following dynamics imposed on the spanning forward Libor rates under the appropriate forward measures,

\[
df_n(t) = (bf_n(t) + (1 - b) f_n(0)) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t; n) \, dW_k^{n+1}(t),
\]

\[
n = 1, \ldots, N - 1.
\]

The following SDEs govern the dynamics of Libor rates under the spot Libor measure,

\[
df_n(t) = (bf_n(t) + (1 - b) f_n(0)) \sqrt{z(t)} \sum_{k=1}^{K} \sigma_k(t; n) \left( \sqrt{z(t)} \mu_k(t, \bar{f}(t); n) \, dt + dW_k(t) \right),
\]

\[
n = 1, \ldots, N - 1.
\]

Here \( \{\mu_k(t, \bar{f}(t); n)\} \) are Libor rate drifts whose form can be found in eg [ABR01].

The shape of the volatility smile produced by the model is primarily governed by the blending parameter \( b \) and the volatility of variance parameter \( \eta \). Blending \( b \) affects the slope of the volatility smile, while \( \eta \) affects its curvature. The mean reversion of variance parameter \( \theta \) controls the evolution of the volatility smile with time, and the volatility structure \( \sigma(\cdot; \cdot) \) determines the overall level of volatilities for all caplets and swaptions. They are usually calibrated to match at-the-money volatilities of all swaptions. For more details see [Sid00] and [AA02].
4.4. Forward rate variances in the forward Libor model with stochastic volatility. In this section we have
\[ \var_{f_n}(T_n) = E_{0}^{n+1} \left( f_n(T_n) - E_{0}^{n+1} f_n(T_n) \right)^2 \]
is evaluated in the model developed in Section 4.3 (recall that these variances are needed for the main valuation formula (4.2)). Let us denote
\[ u_n(s) = \left( \sum_{k=1}^{\lambda} \sigma_k^2(s; n) \right)^{1/2}. \]

**Proposition 4.2.** Denote the moment generating function of \( f_n(T_n) \) by \( \chi_n(\mu) \),
\[ \chi_n(\mu) = E_{0}^{n+1} \left[ \exp \left( \mu \int_0^{T_n} z(s) u_n^2(s) \, ds \right) \right]. \]
Then
\[ \var_{f_n}(T_n) = \frac{f_n^2(0)}{b^2} \left( \chi_n(b^2) - 1 \right). \]

**Proof.** See Appendix A.3. \( \blacksquare \)

Note that the function \( \chi_n(\mu) \) can be computed by numerically solving a Riccati system of ordinary differential equations, see [AP04]. Also note that \( f_n(0) \) enters the expression for the variance (and the right-hand side of (4.2) by extension). The equation (4.2) is of degree 2 in \( f_n(0) \) and can either be solved exactly, or by iteration as explained in Remark 3.1.

The result of Proposition 4.2 is useful from a theoretical prospective, but is not very helpful in practice, since it requires a fully-calibrated stochastic volatility forward Libor model to compute the variances, a model that is typically (relatively) slow to calibrate. The result is simplified in the next section.

4.5. Forward rate variances in a simple stochastic volatility model. Rather than using a full-blown forward Libor model, practitioners prefer reduced-form models to value options on ED futures (as well as caps and swaptions). Since under the appropriate forward measure a forward rate is a martingale, the Black model can be used, and has historically been the model of choice. Relatively recently, more sophisticated models started to appear. Arguably, the most popular among them is the stochastic volatility model.

With the reduced-form model, only a single forward rate at a time is being modeled. For the rate \( f_n(t) \), the following model can for example be used,
\begin{align*}
\frac{dz(t)}{\sqrt{z(t)}} &= \theta(z_0 - z(t)) \, dt + \eta_n \sqrt{z(t)} \, dV(t), \\
\frac{df_n(t)}{f_n(t)} &= \lambda_n \left( b_n f_n(t) + (1 - b_n) f_n(0) \right) \sqrt{z(t)} \, dU(t).
\end{align*}
These equations can be understood to be under the \( T_{n+1} \)-forward measure. The set of parameters \( (\lambda_n, b_n, \eta_n) \) define the volatility smile in options on the rate \( f_n(T_n) \).

While superficially the model (4.6) appears similar to the model (4.4)-(4.5), the key difference is that in the former each forward rate is modeled separately, while in the latter the joint evolution of rates is considered.

The value of the variance in the model (4.6) can be obtained easily.

**Proposition 4.3.** Denote the moment generating function of \( \chi_n^2 \int_0^{T_n} z(s) \, ds \) in the model (4.6) by \( \chi_n^0(\mu) \),
\[ \chi_n^0(\mu) = E_{0}^{n+1} \left[ \exp \left( \mu \lambda_n \int_0^{T_n} z(s) \, ds \right) \right]. \]
Then
\[ \var_{f_n}(T_n) = \frac{f_n^2(0)}{b_n^2} \left( \chi_n^0(b_n^2) - 1 \right). \]

**Proof.** Similar to the proof of Proposition 4.2, see Appendix A.3. \( \blacksquare \)
The moment-generating function \( \chi^0(\mu) \) can be written out in closed form, see Appendix B. Moreover, one of the methods of calibration for the stochastic volatility forward Libor model (see [Pit04]) advocates using the method of matching moment generating functions \( \chi^0(\mu) \) and \( \chi_n(\mu) \) for certain values of \( \mu \). This provides another connection between Proposition 4.2 and 4.3.

We again comment that the expression for the variance \( \text{var}^{n+1} f_n(T_n) \) involves \( f_n(0) \), that would appear on the right-hand side of (4.2) and would make it a quadratic equation.

One should realize that the implied values of parameters \( (\lambda_n, b_n, \eta_n) \) also technologically depend on the parameter \( f_n(0) \). However, the loss of accuracy is negligible if \( (\lambda_n, b_n, \eta_n) \) are implied with the “previous” value of the forward rate \( f_n(0) \), ie the value before the update of the convexity adjustment.

As explained in [AP04], the variance \( \text{var}^{n+1} f_n(T_n) \) can become infinite for certain values of parameters of the model (4.6). In practice, a suitable restriction of the domain of integration will solve this problem.

5. Computing correlations

By direct computation in the forward Libor model (4.5), assuming \( T_j \leq T_m \), we obtain,

\[
\text{corr} \left( f_j(T_j), f_m(T_m) \right) = \mathbb{E}_0 \left( \int_0^{T_j} \varphi \left( f_j(s) \right) \varphi \left( f_m(s) \right) z(s) \sum_{k=1}^{K} \sigma_k(s) \cdot \sigma_k(s;m) \, ds \right) \\
\times \left( \mathbb{E}_0 \left( \int_0^{T_j} \varphi^2 \left( f_j(s) \right) z(s) \sum_{k=1}^{K} \sigma_k^2(s) \, ds \right) \right)^{-1/2} \\
\times \left( \mathbb{E}_0 \left( \int_0^{T_m} \varphi^2 \left( f_m(s) \right) z(s) \sum_{k=1}^{K} \sigma_k^2(s;m) \, ds \right) \right)^{-1/2}.
\]

By “freezing” the forward rates at their initial values when used as arguments in the local volatility functions, a correlation approximation formula\(^6\) is obtained,

\[
(5.1) \quad \text{corr} \left( f_j(T_j), f_m(T_m) \right) = \frac{\int_{0}^{T_j} \sum_{k=1}^{K} \sigma_k(s;j) \cdot \sigma_k(s;m) \, ds}{\left( \int_{0}^{T_j} \sum_{k=1}^{K} \sigma_k^2(s;j) \, ds \right)^{1/2} \left( \int_{0}^{T_m} \sum_{k=1}^{K} \sigma_k^2(s;m) \, ds \right)^{1/2}}.
\]

Model parameters \( \{\sigma_k^2(:,m)\} \) are not directly observable; to extract them, the model needs to be calibrated to market inputs. This is why the power of the fully-calibrated forward Libor model is required. Options on Libor rates (caplets and ED options) do not contain this information; calibrating exclusively to these options is not sufficient to extract the required correlation information from the market. Thus, the calibration procedure needs to include a wider range of instruments. Fortunately, European swaptions (options to enter interest rate swaps) of various expiries and maturities are very liquid. Since European swaptions can be viewed as options on baskets of Libor rates (see Rebonato’s book [Reb98]), their market-quoted implied volatilities can be seen to contain Libor rate correlation information. A calibration of the forward Libor model (see [Sid00]) to caps and swaptions returns \( \{\sigma_k^2(:,m)\} \), from which the required correlations can be computed with the help of (5.1). In essence, the forward Libor model is used as a “device” to extract implied Libor rate correlations from the swaption market. (A similar approach is advocated in [And04] for calibrating a simple interest rate model for pricing Bermudan swaptions.) Since correlations do not change often, this calibration does not have to be performed intra-day.

6. The formula and analysis

Let us present the final formula that we propose to use. Of the three approaches we developed for computing variances, we advocate using the reduced-form stochastic volatility model of Libor rates (Section 4.5). The other two approaches, while perhaps theoretically more appealing for the reasons of model-independence (Section 4.2) or consistency with correlation computations (Section 4.4), are not as well-suited for practical applications.

\( ^6 \) This formula can be interpreted as a leading-term in the expansion of the original formula in a small volatility scaling parameter.
Before stating the result, let us recall the notations. Let \( \{ f_i(0) \}_{i=1}^{N} \) be the (unknown) sequence of forward rates for the tenor structure \( \{ T_n \}_{n=0}^{N} \). Let \( \{ F_n(0) \}_{n=1}^{N-1} \) be the (known) sequence of futures contracts on these rates. For each \( n, n = 1, \ldots, N-1 \), let the triple \( (\lambda_n, b_n, \eta_n) \) be the set of parameters of the model (4.6) implied from market prices of options on the rate \( f_n(T_n) \) (options expiring at \( T_n \) and with different strikes). Finally, let \( \{ \sigma_k(\cdot|n) \}_{k,n} \) be instantaneous factor volatilities of forward rates as obtained from a calibrated forward Libor model (4.5).

**Theorem 6.1.** For each \( n, n = 1, \ldots, N-1 \), the forward rate \( f_n(0) \) can be obtained from the futures \( \{ F_i(0) \}_{i=0}^{n} \) and forward rates for previous periods \( \{ f_i(0) \}_{i=0}^{n-1} \) by solving the following equation,

\[
(6.1) \quad f_n(0) = V(0) \left( 1 + \frac{1}{2} \sum_{j,m=0}^{n} D_{j,m} f_j(0) f_m(0) \left( \chi_n^0 (b_j^0) - 1 \right)^{1/2} \left( \chi_m^0 (b_j^0) - 1 \right)^{1/2} c_{jm} \right),
\]

with

\[
V(0) = \left[ \prod_{i=0}^{n} \frac{1 + \tau_i f_i(0)}{1 + \tau_i F_i(0)} \right] F_n(0),
\]

\[
D_{j,m} = \left( -\frac{\tau_j}{1 + \tau_j F_j(0)} + 1_{(j=n)} \frac{1}{F_n(0)} \right) \left( -\frac{\tau_m}{1 + \tau_m F_m(0)} + 1_{(m=n)} \frac{1}{F_n(0)} \right) + 1_{(j=m)} \frac{\tau_j}{(1 + \tau_j F_j(0))^2} - 1_{(j=n)} \frac{1}{F_n(0)^2},
\]

and

\[
(6.2) \quad c_{jm} = \frac{\int_0^{T_j} \sum_{k=1}^{K} \sigma_k(s;j) \sigma_k(s;m) \, ds}{\left( \int_0^{T_j} \sum_{k=1}^{K} \sigma_k^2(s;j) \, ds \right)^{1/2}} \left( \int_0^{T_m} \sum_{k=1}^{K} \sigma_k^2(s;m) \, ds \right)^{1/2},
\]

\[
\chi_n^0 (\mu) = E_0^{n+1} \left[ \exp \left( \mu \lambda_n^2 \int_0^{T_n} z(s) \, ds \right) \right].
\]

The functions \( \chi_n^0 (\mu) \) can be computed in closed form, see Appendix B.

**Proof.** The proof follows by combining the statement of Corollary 3.4, equations (3.16), (4.1), Proposition 4.3 and equation (5.1). \( \blacksquare \)

**7. Tests**

The formula we have derived for ED convexity adjustments is unique; it is, to the best of our knowledge, the only one that explicitly incorporates volatility smile. Thus, testing it against other “alternatives” is not very illuminating. Instead, we perform the following studies. First, we look at the effects of volatility smile parameters on ED convexity adjustments. Second, we take the Monte-Carlo-computed values of convexity adjustments in a fully-calibrated stochastic volatility forward Libor model as their “true” values, and proceed to test the formula we derived against them. This checks the accuracy of approximations made in deriving the formula.

In all tests, Eurodollar futures with expiries up to 5 years are considered (Eurodollar futures with expiries up to 4 years are considered liquid; after four years, they are typically replaced with swaps as primary interest rate curve building instruments). There are 20 contracts in total. The first is assumed to start in 3 months from the valuation date.

The results are presented in basis points (bp), ie 0.01-ths of a percent. For reference, a bid-ask spread on a typical ED contract is 0.5 basis points.
7.1. **Effect of stochastic volatility parameters on convexity adjustments.** The effects of stochastic volatility parameters (blending parameters \( b_n \) and volatility of variance parameters \( \eta_n \) from Sections 4.4 and 4.5) on ED convexity adjustments is studied. The convexity adjustments are computed by the formula from Theorem 6.1, with the simple stochastic volatility model (4.6) used to compute the variances. The correlations \( \{c_{jm}\} \) in (6.1) are not taken from a forward Libor model as in (6.2), but specified externally, as explained below.

To allow for easy reproducibility of our test results, simulated market conditions are considered. The conditions are chosen to broadly reflect the features of the US interest rate market at the time of writing. The input rates on futures contracts are given by

\[
F_n(0) = 0.01 + 0.04 \times \frac{T_n}{5},
\]

\( n = 1, \ldots, 20. \)

The “market” stochastic volatilities are given by

\[
\lambda_n = 0.5 - 0.35 \times \frac{T_n}{10},
\]

\( n = 1, \ldots, 20. \)

Also,

\[
z_0 = 1, \quad \theta = 0.3.
\]

The correlations \( \{c_{jm}\} \) in (6.1) are computed according to the formula

\[
c_{jm} = \left( \frac{1 - \exp(-2a \min(J, M))}{1 - \exp(-2a \max(J, M))} \right)^{1/2},
\]

\( a = 0.05, \quad j, m = 1, \ldots, 20, \)

(this is a formula for correlations between appropriate forward continuously-compounded rates in a simple one-factor Gaussian model with constant volatility and mean reversion parameter \( a \)).

Convexity adjustments \( F_n(0) - f_n(0), n = 1, \ldots, 20, \) are evaluated using the formula (6.1) for various levels of blending \( b \) and volatility of variance \( \eta \) (flat parameters \( b_n \equiv b, \eta_n \equiv \eta \) are used). In Table 1, convexity adjustments for various contracts (rows) and levels of blending \( b \) (columns) are presented. In Table 2, convexity adjustments for various contracts (rows) and volatilities of variance \( \eta \) (columns) are presented. The results are presented graphically in Figure 1 and Figure 2.

Clearly, the effect of blending and volatility of variance on convexity adjustments is very significant. For back-end contracts (5 years to expiry), the difference between the extreme blending values (\( b = 0.0 \) and \( b = 1.0 \)) is about 20 basis points, a huge difference. The difference is also extremely large, at about 5 basis points, between the models with \( \eta = 0 \) and \( \eta = 100\% \). Clearly, mis-specifying (or not account for) the volatility smile parameters has a very pronounced impact on convexity adjustments.

7.2. **Formula versus Monte-Carlo.** In this section, an interest rate curve and a forward Libor model are specified, and the approximate formula developed in Theorem 6.1 is compared to the “brute-force” Monte-Carlo simulated values of \( F_n(0), n = 1, \ldots, 20. \) Monte-Carlo values are computed by averaging over 250,000 paths.

An interest rate curve used in testing is given by

\[
P(0, t) = \exp\left(-\int_0^t \phi(s) \, ds\right),
\]

\[
\phi(t) = 0.01 + 0.04 \times \frac{t}{5},
\]

\( t \geq 0. \)
A two-factor forward Libor model is used, with
\[
\sigma_1(t;n) \equiv 0.2, \\
\sigma_2(t;n) = 0.5 \times e^{-0.4t} - 0.15.
\]
Stochastic volatility parameters are given by
\[
b = 0.5, \\
\eta = 0.5, \\
\theta = 0.3, \\
z_0 = 1.
\]

The results are presented in Figure 3. Clearly, the agreement between the “true” model value (computed via Monte-Carlo) and the approximation (6.1) is very good, within 0.1 basis point for all contracts.

REFERENCES


APPENDIX A. PROOFS

A.1. Proof of Proposition 3.1. Kennedy and Hunt in [HK00] showed that in the case of discrete margining, futures price \( F(t,T,S) \) for \( t \) in \( [T_i,T_{i+1}) \) is given by
\[
F(t,T,S) = \mathbb{E}_t \left[ \sum_{j=i+2}^{n} (F(T_j,T,S) - F(T_{j-1},T,S)) B^{-1}_{T_j} \right] + \mathbb{E}_t \left[ F(T_{i+1},T,S) B^{-1}_{T_{i+1}} \right].
\]
Assume, by induction, that for all \( k > i \) and for all \( t \) in \( [T_k,T_{k+1}) \), \( \mathbb{E}_t[f(T,T,S)] = F(t,T,S) \). It will be shown that for all \( t \) in \( [T_i,T_{i+1}) \), \( \mathbb{E}_t[f(T,T,S)] = F(t,T,S) \).

The key to prove the statement is to observe that if \( B_t \) is the numeraire of the spot Libor measure, then \( B_{T_j} \) is known at time \( T_{j-1} \), for any \( j \). Using this fact and the induction assumption,
\[
\mathbb{E}_t \left[ \sum_{j=i+2}^{n} (F(T_j,T,S) - F(T_{j-1},T,S)) B^{-1}_{T_j} \right] = \mathbb{E}_t \left[ \sum_{j=i+2}^{n} B^{-1}_{T_j} \mathbb{E}_{T_{j-1}}[(F(T_j,T,S) - F(T_{j-1},T,S))] \right] = 0.
\]
By the same property,
\[
\frac{\mathbb{E}_t \left[ F(T_{i+1}, T, S) B_{T_{i+1}}^{-1} \right]}{\mathbb{E}_t \left[ B_{T_{i+1}}^{-1} \right]} = \mathbb{E}_t \left[ F(T_{i+1}, T, S) \right]
\]
which proves the result.

A.2. Proof of Theorem 3.3. Define
\[
p(y_0, \ldots, y_n) = \prod_{i=0}^{n} \frac{1}{1 + \tau_i y_i} y_n.
\]
Then
\[
\frac{\partial}{\partial y_j} p = -\frac{\tau_j}{1 + \tau_j y_j} \prod_{i=0}^{n} \frac{1}{1 + \tau_i y_i} y_n + 1_{j=n} \prod_{i=0}^{n} \frac{1}{1 + \tau_i y_i}
\]
\[
= p \left( -\frac{\tau_j}{1 + \tau_j y_j} + 1_{j=n} \frac{1}{y_n} \right)
\]
Also,
\[
\frac{\partial}{\partial y_m} \frac{\partial}{\partial y_j} p = \left( \frac{\partial}{\partial y_m} p \right) \left( -\frac{\tau_j}{1 + \tau_j y_j} + 1_{j=n} \frac{1}{y_n} \right)
\]
\[
+ p \left( 1_{j=m} \left( \frac{\tau_j^2}{(1 + \tau_j y_j)^2} - 1_{m=n} \frac{1}{y_n^2} \right) \right)
\]
\[
= p \left( -\frac{\tau_m}{1 + \tau_m y_m} + 1_{m=n} \frac{1}{y_n} \right) \left( -\frac{\tau_j}{1 + \tau_j y_j} + 1_{j=n} \frac{1}{y_n} \right)
\]
\[
+ p \left( 1_{j=m} \left( \frac{\tau_j^2}{(1 + \tau_j y_j)^2} - 1_{m=n} \frac{1}{y_n} \right) \right).
\]
Recall (3.8),
\[
V(\varepsilon) = \prod_{i=0}^{n} \frac{1 + \tau_i f_i(0)}{1 + \tau_i f_i^\varepsilon(T_i)} f_i^\varepsilon(T_n).
\]
Clearly
\[
V(\varepsilon) = \left( \prod_{i=0}^{n} (1 + \tau_i f_i(0)) \right) p(f_0^\varepsilon(T_0), \ldots, f_n^\varepsilon(T_n)).
\]
Obviously, (3.11) follows from (3.5). Moreover,
\[
V'(\varepsilon) = \left( \prod_{i=0}^{n} (1 + \tau_i f_i(0)) \right) \left( \sum_{j=0}^{n} \frac{\partial}{\partial y_j} p(f_0^\varepsilon(T_0), \ldots, f_n^\varepsilon(T_n)) \frac{\partial f_j^\varepsilon(T_j)}{\partial \varepsilon} \right)
\]
\[
= \left( \prod_{i=0}^{n} (1 + \tau_i f_i(0)) \right) \left( \sum_{j=0}^{n} \frac{\partial}{\partial y_j} p(f_0^\varepsilon(T_0), \ldots, f_n^\varepsilon(T_n)) (f_j(t) - F_j(0)) \right)
\]
where (3.6) was used for the second step. Thus,
\[
V'(0) = \left( \prod_{i=0}^{n} (1 + \tau_i f_i(0)) \right) \left( \sum_{j=0}^{n} \frac{\partial}{\partial y_j} p(F_0(0), \ldots, F_n(0)) (f_j(t) - F_j(0)) \right).
\]
Since
\[
\mathbb{E}_0 (f_j(t) - F_j(0)) = 0,
\]
we obtain
\[ E_0 V'(0) = 0, \]
ie (3.12) is proved.

Differentiating (A.2) with respect to \( \varepsilon \) again, we obtain
\[
V''(\varepsilon) = \left( \prod_{i=0}^{n} (1 + \tau_i f_i(0)) \right) \left( \sum_{j,m} \frac{\partial^2}{\partial y_m \partial y_j} p(f^*_0(T_0), \ldots, f^*_n(T_n)) (f_j(t) - F_j(0))(f_m(t) - F_m(0)) \right),
\]
and
\[
V''(0) = \left( \prod_{i=0}^{n} (1 + \tau_i f_i(0)) \right) p(F_0(0), \ldots, F_n(0)) \left( \sum_{j,m} D_{j,m} (f_j(t) - F_j(0))(f_m(t) - F_m(0)) \right),
\]
where we have used (A.1) and the definition (3.14) of \( D_{j,m} \)'s. Simplifying, we obtain
\[
V''(0) = V'(0) \left( \sum_{j,m} D_{j,m} (f_j(t) - F_j(0))(f_m(t) - F_m(0)) \right).
\]
Taking the expected value \( E_0 \) from both sides, (3.12) is obtained. This concludes the proof of the theorem.

A.3. **Proof of Proposition 4.2.** Recall that
\[
u_n(s) = \left( \sum_{k=1}^{K} \sigma_k^2(s;n) \right)^{1/2}.
\]
Denote
\[
dU_n(s) = \frac{1}{u_n(s)} \sum_{k=1}^{K} \sigma_k(s;n) dW_k^{n+1}(t).
\]
It is easy to see that \( dU_n(s) \) is a drift-less Brownian motion under \( P^{n+1} \).

Under \( P^{n+1} \),
\[
f_n(T_n) = \frac{f_n(0)}{b} \left[ \exp \left( b \int_{T_0}^{T_n} \sqrt{z(s)} u_n(s) \, dU_n(s) - \frac{b^2}{2} \int_{T_0}^{T_n} z(s) u_n^2(s) \, ds \right) - (1 - b) \right].
\]
Thus
\[
\text{var}_{n+1} f_n(T_n) = \frac{f_n^2(0)}{b^2} E^{n+1} \left[ \exp \left( b \int_{T_0}^{T_n} \sqrt{z(s)} u_n(s) \, dB_n(s) - \frac{b^2}{2} \int_{T_0}^{T_n} z(s) u_n^2(s) \, ds \right) - 1 \right]^2.
\]
Conditioning on \( z(\cdot) \) we obtain
\[
\text{var}_{n+1} f_n(T_n) = \frac{f_n^2(0)}{b^2} E^{n+1} \left[ \exp \left( b^2 \int_{T_0}^{T_n} z(s) u_n^2(s) \, ds \right) - 1 \right].
\]
The moment-generating function of \( \int_{T_0}^{T_n} z(s) u_n^2(s) \, ds \) is known (see Appendix B) and is easily computable. Let us denote it by \( \chi_n(z) \). Then
\[
\text{var}_{n+1} f_n(T_n) = \frac{f_n^2(0)}{b^2} \left( \chi_n(b^2) - 1 \right).
\]
The proposition is proved.
Appendix B. Laplace Transform of Integrated Stochastic Variance

Recall the definition
\[ \chi_n(\mu) = E_0^{n+1} \left[ \exp \left( \mu \int_0^T z(s) u_n^2(s) ds \right) \right], \]
where the stochastic variance process \( z(\cdot) \) follows (4.6). As explained in [AA02], the function \( \chi_n(\mu) \) can be represented as
\[ \chi_n(\mu) = \exp \left( A_n(0, T) - z_0 B_n(0, T) \right), \]
where the functions \( A(t, T) \), \( B(t, T) \) satisfy the Riccati system of ODEs
\[ A_n'(t, T) - \theta z_0 B_n(t, T) = 0, \tag{B.1} \]
\[ B_n'(t, T) - \theta B_n(t, T) - \frac{1}{2} \eta^2 B_n^2(t, T) + \mu u_n^2(t) = 0, \tag{B.2} \]
with terminal conditions
\[ A_n(T, T) = 0, \]
\[ B_n(T, T) = 0. \]

For the constant-parameter Laplace transform,
\[ \chi^0_n(\mu) = E_0^{n+1} \left[ \exp \left( \mu \lambda_n^2 \int_0^T z(s) ds \right) \right], \]
the system of ODEs can be solved explicitly, to yield
\[ \chi^0_n(\mu) = \exp \left( A^0_n(0, T) - z_0 B^0_n(0, T) \right), \tag{B.3} \]
\[ B^0_n(0, T) = \frac{2\mu \lambda_n^2 (1 - e^{-\gamma T})}{(\theta + \gamma)(1 - e^{-\gamma T}) + 2\gamma e^{-\gamma T}}, \]
\[ A^0_n(0, T) = \frac{2\theta z_0}{n^2} \log \left( \frac{2\gamma}{\theta + \gamma(1 - e^{-\gamma T}) + 2\gamma e^{-\gamma T}} \right) + 2\theta z_0 \frac{\mu \lambda_n^2 T}{\theta + \gamma}, \]
\[ \gamma = \sqrt{\theta^2 - 2\eta^2 \lambda_n^2 \mu}. \]

Appendix C. Tables and Figures
<table>
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<tr>
<th>$T_n$</th>
<th>$\eta = 0.00$</th>
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Table 1. Convexity adjustments (in basis points) for all Eurodollar contracts up to 5 years (rows) for various values of the blending $b$ (columns).

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Table 2. Convexity adjustments (in basis points) for all Eurodollar contracts up to 5 years (rows) for various values of the volatility of variance $\eta$ (columns).
Figure 1. Eurodollar convexity adjustments for contracts with expiries up to 5 years for different values of the blending parameter $b$. 

ED convexity adjustments in basis points

Expiry in years
Figure 2. Eurodollar convexity adjustments for contracts with expiries up to 5 years for different values of the volatility of variance parameter $\eta$. 
Figure 3. Values (left axis) of the Monte-Carlo simulated and approximate Eurodollar convexity adjustments in a stochastic volatility forward Libor model, and the difference (right axis) between the two, in basis points.